Inferring Periodic Variability of Private Market Returns as Measured by σ from the Range of Value (Wealth) Outcomes over Time

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The purpose of this article is to analyze the variability in wealth outcomes of an actual private market portfolio and then express the results in terms directly comparable to the principal risk measure used in the public markets, *viz.* the variability of periodic value as expressed by its standard deviation. The method we use to calculate the periodic standard deviation is necessarily completely different from the conventional public market computation. Although the standard deviation of a typical private market portfolio is measurable directly, just as it is in the public markets, such a straightforward computation may understate, perhaps radically, the actual risk inherent in the private markets.

This potential for an understatement of the risk inherent in private market investing arises because private market values are determined either by appraisal or by reference either to cost or to a round of financing incorporating outside investors (assuming no public float). Even those private market values determined by reference to the public markets, because of a public float, are often discounted arbitrarily to approximate the cost of their characteristic illiquidity. Thus, private market values may show on investors' books as constant or nearly constant over a relatively extended period of time and the standard deviation of periodic value therefore may be unrealistically low as a risk measure.

A number of commentators, analysts, scholars, consultants, and investment firms have attempted to solve this same, seemingly intractable, problem: how to quantify risk in private markets that are notorious for pricing stickiness. For example, there have been several studies of the risk/return characteristics of the private markets that infer the risks by marking private equity transactions to market by comparison to the values of similar public companies.[1] Others, including The Frank Russell Company and Richards & Tierney, have conducted research along the same lines. Merrill Lynch has produced an index of small public stocks intended to emulate the risk characteristics of private investments.[2] All these studies use one or another indirect methods for estimating private market risk by using various public market multiples to infer the values of private market transactions.

What is needed, instead, is a direct computation of private market risk. The method developed in this article is based on the fact that, while periodic valuations may be sticky and thus suspect, final investment outcomes are just that — final. Using the equations derived in Appendix A to analyze the spread of private market portfolio outcomes, it is possible to calculate the periodic risk required to generate such a spread — by working backwards from the known outcome to the unknown interim fluctuations in value that resulted in that outcome.

As a way to introduce this analytical point of view, begin with an absurd (but instructive) example of using outcomes to judge statistical dispersion: a shotgun blast. Exhibits 1 and 2, compare two different shot patterns (looking sideways at a shotgun fired twenty feet into a wall).

Which shotgun do you think has the shorter barrel? The obvious answer, whether derived from watching movies or from some more practical experience, is the one in Exhibit 2. How do we instinctively know this to be true? A shorter barrel controls the shot for a shorter span of time, thus the shot pattern in Exhibit 2 is likely to express randomness in a more exaggerated fashion than the shot pattern of the longer, more-constrained barrel.

In much the same way, individual investments in a stock portfolio tend to express greater randomness

EXHIBIT 1

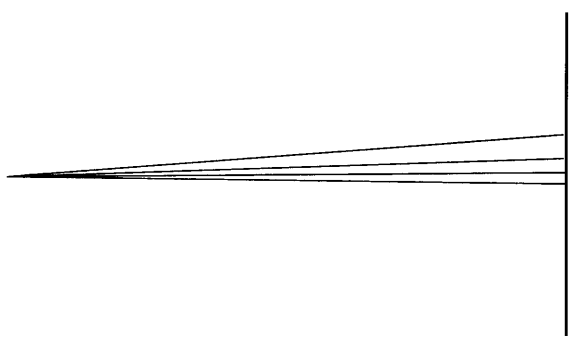
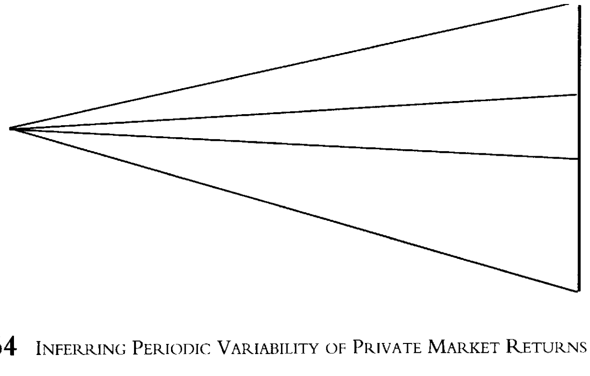


EXHIBIT 2



when they are less-constrained — i.e., assuming no or low correlation, their terminal values exhibit greater dispersion when they are allowed to accumulate risks over time. However, $\sigma_{\bar{r}}$, the average standard deviation of the portfolio return, declines over time as shown in Exhibit 3, because:

$$\sigma_{\bar{r}} = \frac{\sigma}{\sqrt{n}} \tag{1}$$

where n is the number of time periods.

Assuming that $\sigma = 0.211$ and $r = 0.099$, the decrease in $\sigma_{\bar{r}}$ over time results in what appears to be decreased risk to the long-term return of the portfolio, as indicated in Exhibit 4.

While the standard deviation of the portfolio average return declines over time, the terminal wealth to the investor, as indicated by the following equations,[3] continues to diverge over future time periods:

$$W_{max} = (1 + r)^n (1 + \sigma_{\bar{r}})^n \tag{2}$$

indicates the upper range of terminal wealth, where r is the periodic rate of return and n is the number of periods, while

$$W_{min} = \frac{(1 + r)^n}{(1 + \sigma_{\bar{r}})^n} \tag{3}$$

indicates the lower range.

As depicted in Exhibit 5, these two equations taken together describe the relationship of final wealth to original investment over a ten-year time period, a relationship characterized by *increasing* risk as expressed by the uncertainty of the outcome.

These equations depicting final wealth are built upon the assumption that the risk of a portfolio over time can be expressed interchangeably as the periodic risk σ, as the average risk over the time period $\sigma_{\bar{r}}$, or as the probable spread in terminal wealth over the investment's time horizon. Because of the interrelationships among these equations and their common assumptions, it is possible to view risk backwards, as it were, inferring from the spread in outcomes (in terms of final wealth) what the periodic risk of the portfolio must have been.

EXHIBIT 3

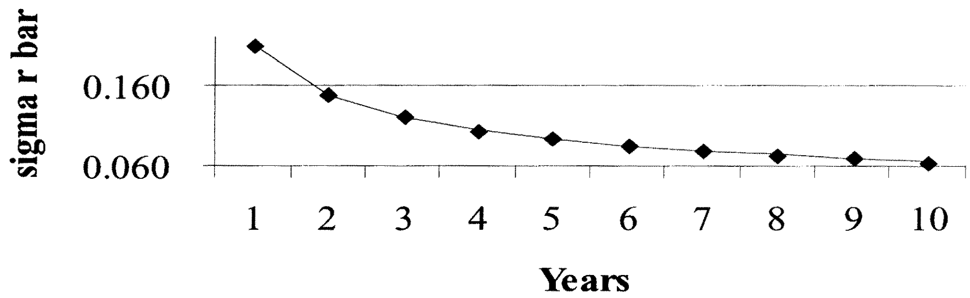


EXHIBIT 4

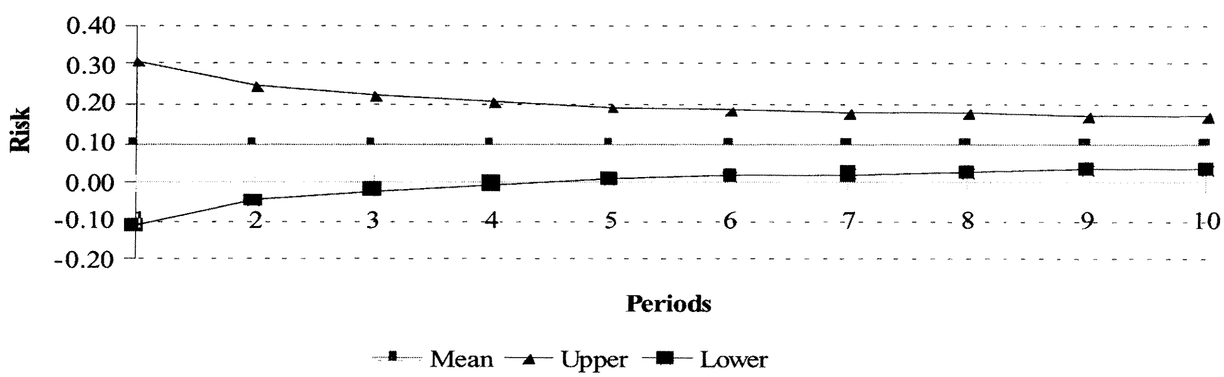
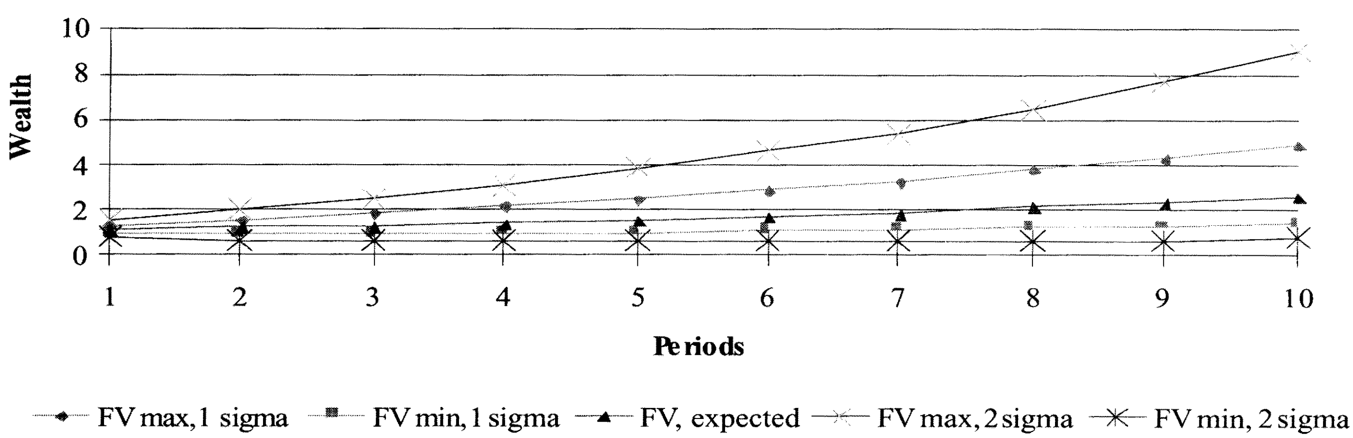


EXHIBIT 5



By a) solving these two equations for r, setting them equal to each other, and then solving for $\sigma_{\bar{r}}$, and b) solving these two equations for $\sigma_{\bar{r}}$, setting them equal to each other, and then solving for r, we can determine the relationship between the spread of wealth outcomes over time and the periodic risk as follows (see Appendix A for a detailed derivation):

$$\sqrt[2n]{W_{max} / W_{min}} - 1 = r \tag{A}$$

$$\sqrt[2n]{W_{max} / W_{min}} - 1 = \sigma_{\bar{r}} \tag{B}$$

In using these two equations, it is important to keep in mind that the wealth amounts involved are those within one standard deviation of the expected value of the terminal wealth of the portfolio, asset class, industry specialization, etc., being analyzed. Thus, the initial analytical task is to determine the mean return-multiple, as well as the standard deviation of the return-multiple, at the time horizon of interest.

Because investment returns must somehow be reinvested in order to understand the provenance of the terminal wealth, the analyst must assume reinvestment in some a vehicle that is sufficiently liquid for the purpose. (We assume a reinvestment in the S&P 500 index.)

NUMERICAL EXAMPLE: 1989, A VINTAGE YEAR

Exhibit 6 contains the spread, in multiples of investment returned, of a portfolio of nine investments made in the 1989 vintage year. The multiple shown in this exhibit signifies the terminal wealth accruing to each dollar employed in the investment, assuming that all distributed returns from each investment are invested in the S&P 500 and then rolled forward to the terminal point. There are nine annual periods from 1989 to 1998, so n = 9. The mean multiple is 5.64, the standard deviation is 3.45. The maximum and minimum wealth in the equations above are at the one σ level (i.e., with a probability of 68%), which means that the maximum for purposes of our analysis is 5.64 + 3.45 = 9.09, while the minimum is 5.64 – 3.45 = 2.19.

EXHIBIT 6

ID	Multiple
777	13.29
767	9.29
765	5.75
718	5.60
759	5.53
775	3.81
763	3.63
757	2.55
740	1.35

It is extremely important to note that the multiples shown in Exhibit 6 are in effect equally weighted. In fact, these investments were (and are) not at all equally weighted, which is why the actual portfolio return is not equal to the return calculated by our Equation (A).[4] Putting all investments in the portfolio on an equal footing however, makes clear the contribution of each to the variability of return, which is the purpose of the calculation.

The calculation of the implied portfolio return is a straightforward application of Equation (13) in Appendix A to this data, as follows:

$$\sqrt[2n]{W_{max} / W_{min}} - 1 = r \qquad \text{thus}$$

$$\sqrt[18]{9.09 \times 2.19} - 1 = r$$

and r = 18.08%.

In the same fashion, the calculation of periodic portfolio risk involves substituting the facts above into equation B and then using the result as input into Equation (1):

$$\sqrt[2n]{W_{max} / W_{min}} - 1 = r \qquad \text{thus}$$

$$\sqrt[18]{9.09 / 2.19} - 1 = \sigma_{\bar{r}} \text{ and } \sigma_{\bar{r}} = 8.23\%.$$

Referring to Equation (1), $\sigma = \sqrt{n}\sigma_{\bar{r}}$, we obtain σ = 3 × 8.23% = 24.68%. Thus, the Sharpe (or information) ratio implied by these calculations is 18.08%/24.68% = 0.7325.

It is extremely interesting to note that the equivalent long-term Sharpe ratio for the S&P 500, according to Ibbotson and Sinquefield's 1987 study, is 12.0%/

21.1% = 0.5687.[5] Given our private market portfolio's superior Sharpe ratio, and assuming that it is representative of the analyst's experience, and it has a low correlation with the S&P 500, these results seem to lead to the conclusion that optimal asset allocation should include a significant investment in the private markets.

Using the example risk/return parameters calculated earlier in conjunction with the parameters contained in the Ibbotson and Sinquefield study of long-term market returns for both stocks and bonds, one could infer that the exact level of correlation between the public and private markets may be close to irrelevant. An empirical test of these example risk/return parameters, using all possible combinations of allocations among stocks, bonds, and private markets investments in 10% increments, reveals that the Markowitz frontier is dominated by the private market asset class whether the private market's correlation with the public market is high, low, or zero.

Consider the efficient frontier shown in Exhibit 7, which is calculated assuming that the correlation between the private market and the public market is the same as the Ibbotson and Sinquefield calculation of the correlation between large-cap stocks and small-cap stocks: i.e., r = 0.82. This implies that R^2 = 67%, meaning that movements in large-cap stocks describe 67% of the movement in small-cap stocks.

The Exhibit 7 datapoint labeled "Best Sharpe" represents an asset allocation of 0% large-cap stocks, 70% bonds, and 30% private equities. While the associated portfolio return of approximately 8.9% might be unacceptably low for most institutional investors, the datapoint associated with a more robust long-term portfolio return of 15.4% utilizes an asset allocation of 0% large-cap stocks, 20% bonds, and 80% private equities.

The Markowitz frontier in Exhibit 8 assumes a much lower correlation coefficient between large-cap stocks and private equities of r = 0.4 (implying that 16% of the movement in private equity values can be explained by movement in the prices of large-cap stocks). This exhibit makes it clear that the effect of correlation on asset allocation is minimal, since the best Sharpe ratio of a poorly correlated portfolio implies a portfolio return of 9.4%, resulting from an allocation of 10% large-cap stocks, 60% bonds, and 30% private equities. An asset allocation of 20% large-cap stocks, 10% bonds, and 70% private equities results in a portfolio return of 15.4% (thus outperforming the long-term S&P 500 index by approximately 540 basis points).

Finally, the Markowitz frontier depicted in Exhibit 9 is calculated with the assumption that large-cap stocks and private equities are completely uncorrelated (i.e., r = 0, implying that none of the movement in private equity values can be explained by price movements in large-cap stocks). This exhibit can be interpreted as follows: if private market returns are uncorrelated with the conventional asset classes, the best Sharpe ratio results in a portfolio return of 9.9%, which is associated with an asset allocation of 20% large-cap stocks, 50% bonds, and 30% private equities. A portfolio return of 15.6% results from an asset allocation of 30% large-cap stocks, 0% bonds, and 70% private equities.

EXHIBIT 7
The Markowitz Frontier (Highly Correlated)

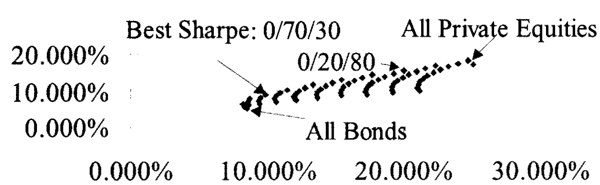


EXHIBIT 8
The Markowitz Frontier (Poorly Correlated)

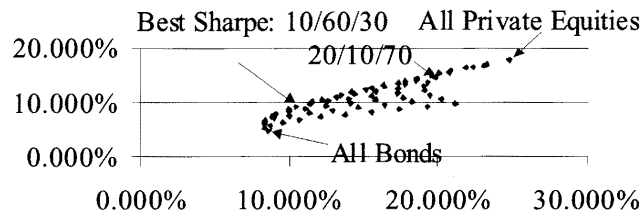


EXHIBIT 9
The Markowitz Frontier (Uncorrelated)

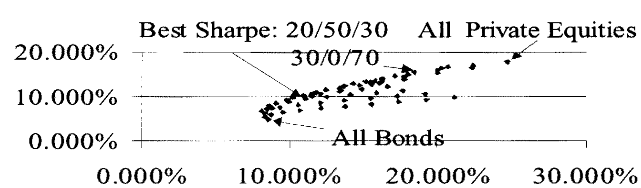


Exhibit 10

Correlation	Best Sharpe	Approx. 15% Return
High	0/70/30	0/20/80
Poor	10/60/30	20/10/70
None	20/50/30	30/0/70

Exhibit 10 summarizes the effect on optimal asset allocation assuming 1) a risk/return profile for private equity as stated above, 2) risk/return profiles for large-cap stocks and bonds as detailed in the 1987 Ibbotson and Sinquefield study, and 3) the correlation of large-cap stocks and private equities assumed in Exhibits 7, 8, and 9.

Given these assumptions and calculations, the correlation of the private markets and large-cap public stocks is essentially irrelevant to the percentage allocated to the private markets needed to attain the highest Sharpe ratio — and it is nearly so when the investment objective is simply to achieve a 15% portfolio return.

SUMMARY

From the chief investment officer's perspective our Equations (A) and (B), as derived in Appendix A, are potentially extremely useful because they make it possible to estimate the periodic risk of a private investment portfolio (as expressed by σ) based on the actual outcomes of its investments. Better knowledge of risk, in particular the ability to assess risk in the private market context in a manner directly comparable to risk assessment in the public markets, should enable one to make better asset allocation decisions and thus to earn better returns for a given portfolio risk profile. The examples we provide imply that this may be true even if the problem of determining the correlation between the public and private markets proves to be intractable.

From the portfolio manager's perspective, it should be possible, given a particular spread in outcomes in the portfolio of an investment manager, to determine the periodic portfolio risk implied by that spread, the effective portfolio return, and the resulting Sharpe ratio. By comparing the Sharpe ratios of various investment managers (possibly grouped by asset class, time period, industry specialization, size, etc.) it should be possible to determine which track records are evidence of prudent levels of risk and which are the result of good fortune. It should also be possible to compare the risk/return characteristics of private equity managers with those of public stock managers on an apples-to-apples basis.

It is important to note, however, that there is one important caveat in using this article's method for calculating periodic risk. Like all statistical methods, insufficient data points can result in incomprehensible or clearly erroneous results. The determination of how many data points are required for meaningful results will be the subject of a future article.

Appendix

Expressing Equation (2) as a log transformation, the upper boundary of terminal wealth in time period n can be expressed in log terms as:

$$\ln W_{max} = n\ln(1+r) + n\ln(1+\sigma_{\bar{r}}) \tag{A-1}$$

Factoring and solving for $\sigma_{\bar{r}}$ in (A-1), we obtain

$$\ln W_{max} = n[\ln(1+r) + \ln(1+\sigma_{\bar{r}})] \tag{A-2}$$

which can be expressed as

$$\frac{\ln W_{max}}{n} - \ln(1+r) = \ln(1+\sigma_{\bar{r}}) \tag{A-3}$$

Expressing Equation (3) in log terms, the lower boundary of terminal wealth in time period n can be expressed and simplified as:

$$\ln W_{min} = n\ln(1+r) - n\ln(1+\sigma_{\bar{r}}) \tag{A-4}$$

which can be simplified to

$$\ln W_{min} = n[\ln(1+r) - \ln(1+\sigma_{\bar{r}})] \tag{A-5}$$

and then expressed as

$$\ln(1+\sigma_{\bar{r}}) = \ln(1+r) - \frac{\ln W_{min}}{n} \tag{A-6}$$

Since we have two equations solved for $\ln(1 + \sigma_{\bar{r}})$, we can compute r, given W_{max} and W_{min}, by making the two equations equal and solving for r as follows:

$$\frac{\ln W_{max}}{n} - \ln(1+r) = \ln(1+r) - \frac{\ln W_{min}}{n} \qquad \text{(A-7)}$$

$$\frac{\ln W_{max}}{n} + \frac{\ln W_{min}}{n} = 2\ln(1+r) \qquad \text{(A-8)}$$

$$\frac{1}{2n}\left(\ln W_{max} + \ln W_{min}\right) = \ln(1+r) \qquad \text{(A-9)}$$

thus, solving for r, we conclude:

$$\sqrt[2n]{W_{max} W_{min}} - 1 = r \qquad \text{(A-10)}$$

Similarly, to determine $\sigma_{\bar{r}}$, we must first solve the W_{max} and W_{min} equations for r, then make the resulting equations equal and solve for $\sigma_{\bar{r}}$. First, solving Equation (3) for r we obtain:

$$\frac{\ln W_{max}}{n} - \ln(1+\sigma_{\bar{r}}) = \ln(1+r) \qquad \text{(A-11)}$$

Solving (A-6) for r we obtain

$$\frac{\ln W_{min}}{n} + \ln(1+\sigma_{\bar{r}}) = \ln(1+r) \qquad \text{(A-12)}$$

Making Equations (A-11) and (A-12) equal, we can solve for $\sigma_{\bar{r}}$:

$$\frac{\ln W_{max}}{n} - \ln(1+\sigma_{\bar{r}}) = \frac{\ln W_{min}}{n} + \ln(1+\sigma_{\bar{r}}) \qquad \text{(A-13)}$$

$$\frac{\ln W_{max}}{n} - \frac{\ln W_{min}}{n} = 2\ln(1+\sigma_{\bar{r}}) \qquad \text{(A-14)}$$

$$\frac{1}{2n}\left(\ln W_{max} - \ln W_{min}\right) = \ln(1+\sigma_{\bar{r}}) \qquad \text{(A-15)}$$

thus, solving for $\sigma_{\bar{r}}$ we obtain:

$$\sqrt[2n]{W_{max} / W_{min}} - 1 = \sigma_{\bar{r}} \qquad \text{(A-16)}$$

ENDNOTES

[1]See Gompers and Lerner [1997].

[2]See Gallante [1993].

[3]Equations (1), (2), and (3) are from Bodie, Kane, and Marcus [1989, pp. 849-850], which cite Merton and Samuelson [1997].

[4]Another important measure of success in a private investment portfolio is an actual return in excess of the return predicted by the equations derived above. An actual return in excess of the predicted return indicates that more money was actually invested in the more successful investments, indicating a positive relationship between the portfolio manager's selections and the overall return. In other words, in terms of performance attribution, the portfolio manager's asset selection contributed positively to the outcome.

[5]Assumes the arithmetic average return of the S&P 500 from 1926 to 1987. The outcome, assuming the time-weighted rate of return of 9.9% over the same period, is a Sharpe ratio of 0.4714.

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